第五章、乘积空间

- §5.1 有限维乘积空间(略)
- ~~§5.2 多维L-S 测度~~
- §5.3 可列维乘积空间
- §5.4 任意维乘积空间

- **乘积空间** $X_1 \times \cdots \times X_n$, 矩形$\prod_{k=1}^{n} A_k$. e.g.

$$X = \prod_{k=1}^{n} X_k := \{\vec{x} = (x_1, \cdots, x_n) : x_k \in X_k, k = 1, \cdots, n\}.$$

- **可测结构**. 设$\mathscr{F}_k$ 是X_k 上的σ 代数, $(k = 1, \cdots, n)$.
- 可测矩形:

$$\mathscr{Q} := \{\star : A_k \in \mathscr{F}_k, k = 1, \cdots, n\}.$$

- 命题5.1.1. $\mathscr{Q}$ 是半环, 且$X \in \mathscr{Q}$.
- 乘积σ 代数,

$$\mathscr{F} = \mathscr{F}_1 \times \cdots \times \mathscr{F}_n = \prod_{k=1}^{n} \mathscr{F}_k := \sigma(\mathscr{Q}).$$

- **投影**: $\pi_k : (x_1, \cdots, x_n) \mapsto x_k$.
- 命题5.1.2. (1) $\pi_k : (X, \mathscr{F}) \to (X_k, \mathscr{F}_k)$, 是可测映射;

  (2) $\mathscr{F} = \sigma\left(\bigcup_{k=1}^n \pi_k^{-1}\mathscr{F}_k\right)$

  $$= \sigma\Big(\big\{\{\vec{x} : x_k \in A_k\} : 1 \leqslant k \leqslant n, A_k \in \mathscr{F}_k\big\}\Big).$$
- 定理5.1.3. 设$f = (f_1, \cdots, f_n) : \Omega \to X$.

  则$f : (\Omega, \mathscr{S}) \to (X, \mathscr{F})$ 可测当且仅当f_k 可测, $\forall k$.
- **截口**. 设$A \in \mathscr{F}$, $f : (X, \mathscr{F}) \to (Y, \mathscr{S})$.

  $$A|_{x_1, \cdots, x_i} = \{(x_{i+1}, \cdots, x_n) : (x_1, \cdots, x_n) \in A\}.$$

  $$f|_{x_1, \cdots, x_i} : (x_{i+1}, \cdots, x_n) \mapsto f(x_1, \cdots, x_n).$$
- 定理5.1.4. $A|_{x_1, \cdots, x_i}$, $f|_{x_1, \cdots, x_i} \in \prod_{k=i+1}^n \mathscr{F}_k$.

- **测度**. 例, 联合= 边缘× 条件,

$$P(\xi \in A_1, \eta \in A_2) = \int_{A_1} \int_{A_2} p_{\eta|\xi}(x_2|x_1)\mathrm{d}x_2 p_\xi(x_1)\mathrm{d}x_1$$
$$= \int_{A_1} \mu_\eta(A_2|\xi = x_1)\mu_\xi(\mathrm{d}x_1).$$

- 定义5.1.1. $p(\cdot,\cdot) : (X_1, \mathscr{F}_2) \to \mathbb{R}$, $(x_1, A_2) \mapsto p(x_1, A_2)$ 为从$(X_1, \mathscr{F}_1)$ 到$(X_2, \mathscr{F}_2)$ 的转移函数, 指:
  (1) 固定$x_1 \in X_1$, $p(x_1, \cdot)$ 是$\mathscr{F}_2$ 上的测度;
  (2) 固定$A \in \mathscr{F}_2$, $p(\cdot, A)$ 是$(X_1, \mathscr{F}_1)$ 上的可测函数.

- 若$X_2 = \sum_n A_n$ 且$p(x, A_n) < \infty$, $\forall x \in X_1$, $\forall n$, 则称$p(\cdot,\cdot)$ 为σ 有限的. (强调: 划分与x 无关.)

- 若$p(x, X_2) < \infty$, $\forall x \in X_1$, 则称$p(\cdot,\cdot)$ 为有限的转移函数.
  若$p(x, X_2) = 1$, $\forall x \in X_1$, 则称$p(\cdot,\cdot)$ 为概率转移函数.

- 记$(X,\mathscr{F}) := (X_1\times X_2, \mathscr{F}_1\times\mathscr{F}_2)$, $\hat{X} = X_2\times X_1$.
- 定理5.1.6. 设$p(\cdot,\cdot)$ 为从$(X_1,\mathscr{F}_1)$ 到$(X_2,\mathscr{F}_2)$ 的σ 有限的转移函数. (1) & (3) $\forall X_1$ 上的σ 有限的测度μ_1, $\exists! X$ 上σ 有限的测度μ 使得

$$\mu(A_1\times A_2) = \int_{A_1} p(x_1,A_2)\mu_1(\mathrm{d}x_1);$$

(2) 若$(X,\mathscr{F},\mu)$ 上的可测函数f 的积分存在, 则

$$\int_X f\mathrm{d}\mu = \int_{X_1}\mu_1(\mathrm{d}x_1)\int_{X_2} f(x_1,x_2)p(x_1,\mathrm{d}x_2).$$

- 定理5.1.7(乘积测度, Fubini定理). Esp, $p(x_1,\cdot)\equiv\mu_2$. 此时将μ 记为$\mu_1\times\mu_2$. 若f 的积分存在, 则

$$\int_X f\mathrm{d}\mu_1\times\mu_2 = \int_{X_1}\mu_1(\mathrm{d}x_1)\int_{X_2} f(x_1,x_2)\mu_2(\mathrm{d}x_2)$$

$$= \int_{\hat{X}}\hat{f}\mathrm{d}\mu_2\times\mu_1 = \int_{X_2}\mu_2(\mathrm{d}x_2)\int_{X_1} f(x_1,x_2)\mu_1(\mathrm{d}x_1).$$

- $X=\prod_{k=1}^n X_k$, $\mathscr{F}=\prod_{k=1}^n \mathscr{F}_k$.
- 定理5.1.9(归纳法). (1) 设p_k 为从$(\prod_{i=1}^{k-1} X_i,\prod_{i=1}^{k-1}\mathscr{F}_i)$ 到$(X_k,\mathscr{F}_k)$ 的σ 有限的转移函数. $\forall X_1$ 上(σ 有限的)测度μ_1, $\exists! X$ 上(σ 有限的)的测度μ 使得

$$\mu(\prod_{k=1}^n A_k)=\int_{A_1}\mu_1(\mathrm{d}x_1)\int_{A_2}p_2(x_1,\mathrm{d}x_2)\cdots \int_{A_n}p_n(x_1,\cdots,x_{n-1},\mathrm{d}x_n);$$

(2) 若f 为$(X,\mathscr{F},\mu)$ 上的可测函数且积分存在, 则

$$\int_X f\mathrm{d}\mu=\int_{X_1}\mu_1(\mathrm{d}x_1)\int_{X_2}p_2(x_1,\mathrm{d}x_2)\cdots \int_{X_n}p_n(x_1,\cdots,x_{n-1},\mathrm{d}x_n)f(\vec{x}).$$

- 定理5.1.10(乘积测度). Esp. $p_k(x_1,\cdots,x_{k-1};\cdot)\equiv\mu_k$.

§5.3 可列维乘积空间的概率测度

- 一列可测空间: $(X_n, \mathscr{F}_n)$, $n = 1, 2, \cdots$
- $X = \prod_{n=1}^{\infty} X_n := \{\vec{x} = (x_1, x_2, \cdots) : x_n \in X_n, \forall n\}$.
- 投影: $\pi_n : X \to X_n, \quad \pi_{(n)} : X \to X_{(n)} := \prod_{k=1}^{n} X_k$.
- 有限维: 在$X_{(n)}$ 上, $\mathscr{F}_{(n)} := \prod_{k=1}^{n} \mathscr{F}_k = \sigma(\mathscr{Q}_{(n)})$,

  $\mathscr{Q}_{(n)} := \{\prod_{k=1}^{n} A_k : A_k \in \mathscr{F}_k, k = 1, \cdots, n\}$.
- 用$\pi_{(n)}$ 嵌入X. 有限维可测(矩形)柱集:

  $\mathscr{Q}_{[n]} := \left\{Q_{(n)} \times \prod_{k=n+1}^{\infty} X_k : Q_{(n)} \in \mathscr{Q}_{[n]}\right\} = \pi_{(n)}^{-1} \mathscr{Q}_{(n)}$;

  $\mathscr{F}_{[n]} := \left\{A_{(n)} \times \prod_{k=n+1}^{\infty} X_k : A_{(n)} \in \mathscr{F}_{(n)}\right\} = \pi_{(n)}^{-1} \mathscr{F}_{(n)}$.
- 命题5.3.1. $\mathscr{Q} = \bigcup_{n=1}^{\infty} \mathscr{Q}_{[n]}$ 是半环, 且$X \in \mathscr{Q}$;

  $\mathscr{A} = \bigcup_{n=1}^{\infty} \mathscr{F}_{[n]}$ 是代数;

  $\mathscr{F} = \prod_{n=1}^{\infty} \mathscr{F}_n := \sigma(\mathscr{Q}) = \sigma(\mathscr{A}) = \sigma(\{\pi_n, \forall n\})$.

定理 (定理5.3.2, Tulcea 定理)

设$p_k : (\prod_{i=1}^{k-1} X_i, \prod_{i=1}^{k-1} \mathscr{F}_i) \to (X_k, \mathscr{F}_k)$ 为概率转移函数, $k \geqslant 2$. 则$\forall (X_1, \mathscr{F}_1)$ 上的概率P_1, $\exists ! (\prod_{k=1}^{\infty} X_k, \prod_{k=1}^{\infty} \mathscr{F}_k)$ 上的概率P 使

$$P\left(\prod_{k=1}^{n} A_k \times \prod_{k=n+1}^{\infty} X_k\right)$$

$$\overset{(\star)}{=} \int_{A_1} P_1(\mathrm{d}x_1) \int_{A_2} p_2(x_1, \mathrm{d}x_2) \cdots \int_{A_n} p_n(x_1, \cdots, x_{n-1}, \mathrm{d}x_n).$$

(1) $(\star)$ 定义$\mathscr{F}_{(n)} = \sigma(\mathscr{Q}_{(n)})$ 上的概率(定理5.1.9), ✓;

(2) 利用$\pi_{(n)}$ 得到$\mathscr{F}_{[n]} = \pi_{(n)}^{-1} \mathscr{F}_{(n)}$ 上的概率P_n, ✓;

(3) 证明$P_n = P_{n+1}|_{\mathscr{F}_{[n]}}$, 故得到$\mathscr{A} = \bigcup_{n=1}^{\infty} \mathscr{F}_{[n]}$ 上的函数P.

(4) 证明P 是$\mathscr{A}$ 上的概率, 从而惟一地扩张到$\mathscr{F} = \sigma(\mathscr{A})$ 上.

(4) 证明$P_n = P_{n+1}|_{\mathscr{A}_{[n]}}$.

- $\forall A \in \mathscr{F}_{[n]} = \pi_{(n)}^{-1}\mathscr{F}_{(n)}$, $\exists! A_{(n)} \in \mathscr{F}_{(n)}$ 使得$A = \pi_{(n)}^{-1}A_{(n)}$. 则

$$P_n(A) := \int_{X_1} P_1(\mathrm{d}x_1)\int_{X_2} p_2(x_1, \mathrm{d}x_2)\cdots$$

$$\int_{X_n} \mathbf{I}_{A_{(n)}}(x_1, \cdots, x_n)p_n(x_1, \cdots, x_{n-1}, \mathrm{d}x_n).$$

- $\exists! A_{(n+1)} \in \mathscr{F}_{(n+1)}$ 使得$A = \pi_{(n+1)}^{-1}A_{(n+1)}$.
- $A_{(n+1)} = A_{(n)} \times X_{n+1}$: $\pi_{(n)} = \pi_{(n+1)\to(n)} \circ \pi_{(n+1)}$, 故

$$\pi_{(n)}^{-1}A_{(n)} = \pi_{(n+1)}^{-1} \circ \pi_{(n+1)\to(n)}^{-1}A_{(n)} = \pi_{(n+1)}^{-1}(A_{(n)} \times X_{n+1}).$$

- $P_{n+1}(A) = P_n(A)$: $**$ $=$

$$\int_{X_{n+1}} \mathbf{I}_{A_{(n)}\times X_{n+1}}(x_1, \cdots, x_n; x_{n+1})p_{n+1}(x_1, \cdots, x_n, \mathrm{d}x_{n+1}).$$

(5) 证明P 是$\mathscr{A}$ 上的概率.

- P 具有可加性. 只需证明P 在$\varnothing$ (上)连续:

$$\text{若} A_{[1]}, A_{[2]}, \cdots \in \mathscr{A},\ A_{[n]} \downarrow \varnothing,\ \text{则} P(A_{[n]}) \to 0.$$

- 理由(定理2.1.6): 设$A_1, A_2, \cdots, A := \sum_{n=1}^{\infty} A_n \in \mathscr{A}$, 则

$$A_{[n]} = \sum_{i=n+1}^{\infty} A_i = A - \sum_{i=1}^{n} A_i \in \mathscr{A},\ P(A_{[n]}) = P(A) - \sum_{i=1}^{n} P(A_i).$$

- 反证法验证$\star$: 谬设 $P(A_{[n]}) \nrightarrow 0$, 则$\varepsilon :=\downarrow \lim\limits_{n\to\infty} P(A_{[n]}) > 0$.
- 存在$1 \leqslant m_1 < m_2 < \cdots$ 使得$A_{[n]} \in \mathscr{F}_{[m_n]}$, $\forall n$.

  不妨设$m_n = n$, 否则考虑$B_{[k]} := A_{[n]}$, $m_n \leqslant k < m_{n+1}$.
- $\exists! A_{(n)}$ 使得$A_{[n]} = \pi_{(n)}^{-1} A_{(n)}$, 则

$$A_{[n]} = \pi_{(n+1)}^{-1}(A_{(n)} \times X_{n+1}) \supseteq A_{[n+1]} \Rightarrow A_{(n)} \times X_{n+1} \supseteq A_{(n+1)}.$$

(5) 证明P 是$\mathscr{A}$ 上的概率(续).

- $A_{(n)} \times X_{n+1} \supseteq A_{(n+1)}$. 等价地,

$$\mathbf{I}_{A_{(n+1)}}(x_1, \cdots, x_{n+1}) \leqslant \mathbf{I}_{A_{(n)}}(x_1, \cdots, x_n).$$

- $0 \leqslant \phi_{1,n+1}(x_1) \leqslant \phi_{1,n}(x_1) \leqslant 1$, 其中$\phi_{1,n}(x_1) :=$

$$\int_{X_2} p_2(x_1, \mathrm{d}x_2) \cdots \int_{X_n} \mathbf{I}_{A_{(n)}}(x_1, \cdots, x_n) p_n(x_1, \cdots, x_{n-1}, \mathrm{d}x_n).$$

- 注: $P(A_{[n]}) = P_n(A_{[n]}) = \int_{X_1} \phi_{1,n}(x_1) P_1(\mathrm{d}x_1)$.
- 令$\phi_1 :=\downarrow \lim_{n\to\infty} \phi_{1,n}$, 则由DCT,

$$\int_{X_1} \phi_1 \mathrm{d}P_1 = \downarrow \lim_{n\to\infty} \int_{X_1} \phi_{1,n} \mathrm{d}P_1 = \varepsilon > 0, \quad (\text{根据谬设}).$$

(5) 证明P 是$\mathscr{A}$ 上的概率(续).

- $\exists \tilde{x}_1 \in X_1$ 使得$\phi_1(\tilde{x}_1) > 0$.
- $\tilde{x}_1 \in A_{(1)}$. 否则

$$\mathbf{I}_{A_{(n)}}(\tilde{x}_1, x_1, \cdots, x_n) \leqslant \mathbf{I}_{A_{(1)}}(\tilde{x}_1) = 0 \Rightarrow \phi_{1,n}(\tilde{x}_1) = 0, \forall n.$$

- $\phi_2(x_2) =\downarrow \lim_{n\to\infty} \phi_{2,n}(x_2)$, 其中$\phi_{2,n}(x_2) :=$

$$\int_{X_3} p_3(\tilde{x}_1, x_2, \mathrm{d}x_3) \cdots \int_{X_n} \mathbf{I}_{A_{(n)}}(\tilde{x}_1, x_2, \cdots, x_n) p_n(\tilde{x}_1, x_2, \cdots, x_{n-1}, \mathrm{d}x_n).$$

- $\exists \tilde{x}_2$ 使得$(\tilde{x}_1, \tilde{x}_2) \in A_{(2)}$ 使得$\phi_2(\tilde{x}_2) > 0$. $\cdots\cdots$
- $(\tilde{x}_1, \tilde{x}_2, \cdots) \in \bigcap_{n=1}^{\infty} A_{[n]} = \varnothing$, 矛盾!
- 故$P(A_{[n]}) \to 0$, 即P 在$\varnothing$ (上)连续.

定理 (定理5.3.3, Kolmogorov 定理)

设P_k 为$(X_k, \mathscr{F}_k)$ 上的概率，则$\exists!(\prod_{k=1}^{\infty} X_k, \prod_{k=1}^{\infty} \mathscr{F}_k)$ 上的测度P 使得

$$P\left(\prod_{k=1}^{n} A_k \times \prod_{k=n+1}^{\infty} X_k\right) = \prod_{k=1}^{n} P_k(A_k).$$

- 证：取$p_k : (\prod_{i=1}^{k-1} X_i, \prod_{i=1}^{k-1} \mathscr{F}_i) \to (X_k, \mathscr{F}_k)$ 为P_k 即可.

§5.4 任意无穷维乘积空间的概率测度

- 一族非空集合$\{X_t, t\in T\}$. 其中, 指标集T 不可数.
- **乘积空间**. $X = \prod_{t\in T} X_t := \{\{x_t, t\in T\} : x_t \in X_t, t\in T\}$.
- 注: 若$X_t \equiv X_0$, 则$x : T \to X_0$ 为T 上取X_0 值的“函数”.
- **投影/限制**. $\forall t\in T$, $\forall U \subseteq S \subseteq T$,

$$\pi_t : X \to X_t, \quad x \mapsto x_t;$$

$$\pi_S : X \to X_S = \prod_{t\in S} X_t, \quad x \mapsto x_S := \{x_t, t\in S\};$$

$$\pi_{S\to U} : X_S \to X_U, \quad \pi_{S\to U}\circ \pi_S = \pi_U.$$

- **可测结构**. 设$\mathscr{F}_t$ 是X_t 上的σ 代数, $\forall t \in T$.
- 有限维可测(矩形)柱集: $|S| < \infty$.

$$\mathscr{Q}_S = \left\{ \pi_S^{-1}\left(\prod_{t\in S} A_t\right), A_t \in \mathscr{F}_t, \forall t \in S \right\}; \quad \mathscr{F}_S = \sigma(\mathscr{Q}_S).$$

- 命题5.4.1 & 5.4.2. $\mathscr{Q}_S$, $\mathscr{Q} = \bigcup_{S\subseteq T,|S|<\infty} \mathscr{Q}_S$ 是半环且$\ni X$.
- 命题5.4.3. $\mathscr{A} := \bigcup_{S\subseteq T,|S|<\infty} \mathscr{F}_S$ 是代数且$\supseteq \mathscr{Q}$.
- 命题5.4.4. $\mathscr{F} := \sigma(\mathscr{Q}) = \sigma(\mathscr{A})$, 且

$$\mathscr{F} = \sigma(\{\pi_t, t \in T\}) = \{\pi_S^{-1} A : A \in \mathscr{F}_S, S\text{可列}\}.$$

- 注1: $\star := \sigma\left(\bigcup_{t\in T} \pi_t^{-1}\mathscr{F}_t\right)$.
- 注2: 证明$\mathscr{F}_0 = \{\pi_S^{-1} A : A \in \mathscr{F}_S, S\text{可列}\}$ 是σ 代数.

- **随机过程**. $(\Omega, \mathscr{S})$ 为(样本)可测空间.
  指标集T 视为“时间”. 时刻$t \in T$, 取值空间为$(X_t, \mathscr{F}_t)$.
- $f_t : \Omega \to X_t, \forall t \in T$.
  $f := \{f_t, t \in T\}$ 视为$\Omega \to X := \prod_{t \in T} X_t$ 的映射.
- 定理5.4.5. 记$\mathscr{F} = \prod_{t \in T} \mathscr{F}_t$. 则

$$f : (\Omega, \mathscr{S}) \to (X, \mathscr{F}) \text{ iff } f_t : (\Omega, \mathscr{S}) \to (X_t, \mathscr{F}_t), \ \forall t \in T.$$

- 若$(X_t, \mathscr{F}_t) \equiv (S, \mathscr{S}_0)$, 则称$f$ 是随机过程; 称S 为(取)值空间/状态空间; 称$f(\omega) = \{f_t(\omega) : t \in T\} \in S^T$ 为轨道.

(1) $\forall n \geqslant 1$, $t_1, \cdots, t_n \in T$ 互不相同, $P_{t_1,\cdots,t_n}$ 是$\mathscr{B}_{\mathbb{R}}^n$ 上的概率.

(2) 对$t_1, \cdots, t_n$ 的任意重排$t(1), \cdots, t(n)$,

$$P_{t_1,\cdots,t_n}\left(\prod_{i=1}^n A_{t_i}\right) = P_{t(1),\cdots,t(n)}\left(\prod_{i=1}^n A_{t(i)}\right),$$

(3) $P_{t_1,\cdots,t_n,t_{n+1}}\left(\prod_{i=1}^n A_{t_i} \times \mathbb{R}\right) = \star\star$.

- 定义5.4.1. 若$\mathbf{P} := \{P_{t_1,\cdots,t_n} : n \geqslant 1, t_1, \cdots, t_n\}$ 满足以上三条, 则称它是相容的.
- 由(2) 可定义P_S, $\forall S \subseteq T$, $|S| < \infty$.
- 由(3), $|S| < \infty$, $U \subseteq S \Rightarrow P_S \circ \pi_{S\to U}^{-1} = P_U$.

定理 (定理5.4.8, Kolmogorov 定理)

若**P** 是相容的，则$\exists!(\mathbb{R}^T, \mathscr{B}_{\mathbb{R}}^T)$ 上的概率P，使得

$$P(\pi_S^{-1}A) = P_S(A), \quad \forall |S| < \infty, A \in \mathscr{B}_{\mathbb{R}}^S.$$

- $\mathscr{F}_0 = \{\pi_{T_0}^{-1}(A) : A \in \mathscr{F}_{T_0}, T_0\text{可列}\}$.
- Step 1. (引理5.4.6). 固定可列的$T_0 \subseteq T$，定义$P(\pi_{T_0}^{-1}A)$.

  将T_0 中的指标编号: $t_1, t_2, \cdots$，视为$1, 2, \cdots$.

  由$n \to n+1$ 相容，可产生正则条件概率，此即转移概率测度.

  用Tulcea 定理，$\checkmark$.
- Step 2.1. (引理5.4.7). 对不同编号良定. 由重排相容，$\checkmark$.
- Step 2.2. 对不同的T_0 良定:

  T_1, T_2 可列，$\pi_{T_1}^{-1}A_1 = \pi_{T_2}^{-1}A_2$. 则$P_{T_1}(A_1) = P_{T_2}(A_2)$.

Step 2.2. 对不同的T_0 良定.

$$T_1, T_2 \text{ 可列}, \pi_{T_1}^{-1} A_1 = \pi_{T_2}^{-1} A_2. \text{ 则} P_{T_1}(A_1) = P_{T_2}(A_2).$$

- $T_0 = T_1 \cup T_2$ 可列. 例, $\pi_{T_0 \to T_1} \circ \pi_{T_0} = \pi_{T_1}$, 故

$$\pi_{T_1}^{-1} A_1 = \pi_{T_0}^{-1}(\pi_{T_0 \to T_1}^{-1} A_1) \triangleq \pi_{T_0}^{-1} A_0, \quad \text{其中}, A_0 = \pi_{T_0 \to T_1}^{-1} A_1.$$

- 只需证明$P_{T_0}(A_0) = P_{T_1}(A_1)$, 其中$A_0 = \pi_{T_0 \to T_1}^{-1} A_1$, 即,

$$P_{T_0} \circ \pi_{T_0 \to T_1}^{-1} = P_{T_1}, \qquad (\text{在}\mathscr{F}_{T_1}) \text{ 上.})$$

- $\mathscr{Q}_{T_1} := \bigcup_{S \subseteq T_1, |S| < \infty} \mathscr{Q}_S$ 是半环, 且$\sigma(\mathscr{Q}_{T_1}) = \mathscr{F}_{T_1}$;

  LHS 与RHS 都是概率测度, 在$\mathscr{Q}_{T_1}$ 上相等, 故✓.

Step 3. P 为概率.

- 设$A_1, A_2, \cdots \in \mathscr{B}_{\mathbb{R}}^{T}$, 两两不交.
- $\exists T_n \subseteq T$ 可列, $B_n \in \mathscr{B}_{\mathbb{R}}^{T_n}$ 使得$A_n = \pi_{T_n}^{-1} B_n$.
- 令$T_0 = \bigcup_{n=1}^{\infty} T_n$. P_{T_0} 是$\mathscr{F}_{T_0}$ 上的概率, 故$\checkmark$.

定理 (定理5.4.9)

$(\mathbb{R}, \mathscr{B}_{\mathbb{R}}, P_t)$, $t \in T$ 是概率空间. 则$\exists!(\mathbb{R}^T, \mathscr{B}_{\mathbb{R}}^T)$ 上的概率P, 使得

$$P(\{x : x_{t_i} \in A_i, i = 1, \cdots, n\}) = \prod_{i=1}^{n} P_{t_i}(A_i).$$

- 注: 定理中的P 记为$\prod_{t \in T} P_t$.