RESEARCH STATEMENT
Characteristic classes and ε-factors for constructible étale sheaves

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Contents

Overview. My research focuses on geometric ramification theory for constructible étale sheaves and its motivic counterpart. In the past five years (2019-2023), I have published three peer-reviewed articles [UYZ20, YZ21, JY21] and four preprints [YZ22, JY22, JSY22, XY23] with my collaborators. The main achievements of these articles are as follows:

(1) In [UYZ20], we prove a twist formula for the ε-factor of a constructible sheaf, which is conjectured[1] by Kato and Saito in [KS08, Conjecture 4.3.11].
(2) In [YZ21], we propose a relative version of Kato-Saito's twist formula. As an evidence of this conjecture, we generalize the cohomological characteristic class defined by Abbes and Saito to a relative case under certain transversality conditions. This notion is further generalized to universal local acyclicity (ULA) sheaves by Lu and Zheng in [LZ22].
(3) In [YZ22], we construct a cohomological characteristic class (called non-acyclicity class) supported on the non-acyclicity locus. Using this class, we confirm the quasi-projective case of Saito's conjecture [Sai17], namely that the cohomological characteristic classes defined by Abbes and Saito can be computed in terms of the characteristic cycles. As other applications, we prove cohomological analogs of the Milnor formula and the conductor formula for constructible sheaves on (not necessarily smooth) varieties.
(4) In [XY23], we define the geometric counterpart of the non-acyclicity class and formulate a Milnor-type formula for non-isolated singularities, which says that the non-acyclicity classes can be calculated in terms of the characteristic cycles.
(5) In the papers [JY21, JY22, JSY22], we study ramification theory for motives. We propose a quadratic version of the Artin conductor for **SH** motives and then construct a quadratic version of the Grothendieck-Ogg-Shafarevich formula.

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[1] The original conjecture uses Swan class, but in our paper, we replace it with characteristic class.

This research statement is organized as follows. In Section 1, we give a quick overview of geometric ramification theory. Section 2 introduces our work on Kato-Saito's conjecture on the twist formula of ε-factors. In Section 3, we summarize the properties of non-acyclicity classes and discuss Saito's conjecture on characteristic classes. In Section 4, we recall the construction of non-acyclicity classes. In Section 5, we present a review of our work on ramification theory for motives.

1. Ramification theory

In this section, we provide a brief overview of ramification theory, concentrating specifically on the discussion of characteristic classes and characteristic cycles for constructible étale sheaves due to personal constraints.

1.1. Let S be a Noetherian scheme and Sch_S the category of separated schemes of finite type over S. Let Λ be a finite local ring such that the characteristic of its residue field is invertible on S. For any scheme $X \in \mathrm{Sch}_S$, we denote by $D_{\mathrm{ctf}}(X, \Lambda)$ the derived category of complexes of Λ-modules of finite tor-dimension with constructible cohomology groups on X.

1.2. Consider the following assumptions on S:

(G) S is the spectrum of a perfect field k of characteristic $p > 0$.
In this geometric case, we have a well-defined number $\dim\mathcal{K} = \mathrm{rank}\mathcal{K}$ for $\mathcal{K} \in D_{\mathrm{ctf}}(S, \Lambda)$.

(A) S is the spectrum of a discrete valuation ring. Let η be the generic point of S and s its closed point. We assume that the residue field $k(s)$ is a perfect field of characteristic $p > 0$.
In this arithmetic case, for any $\mathcal{K} \in D_{\mathrm{ctf}}(S, \Lambda)$, we have the Swan conductor $\mathrm{Sw}\mathcal{K}$ (measuring the wild ramification), the total dimension $\mathrm{dimtot}\mathcal{K} = \dim\mathcal{K}_{\overline{\eta}} + \mathrm{Sw}\mathcal{K}$ and the Artin conductor $a_S(\mathcal{K}) = \dim\mathcal{K}_{\overline{\eta}} - \dim\mathcal{K}_{\overline{s}} + \mathrm{Sw}\mathcal{K} = \mathrm{dimtot}R\Phi_{\mathrm{id}}(\mathcal{K})$, where $R\Phi$ is the vanishing cycles functor.

In ramification theory, there exist three distinct versions of higher-dimensional analogues of Artin/Swan conductors: the cohomological characteristic class introduced by Abbes and Saito in [AS07], the Swan class presented by Kato and Saito in [KS08, KS12], and the characteristic class/cycle constructed by Saito in [Sai17] based on Beilinson's singular support. These classes are related to the following Riemann-Roch type questions:

Question 1.3. Let $f : X \to S$ be a separated morphism of finite type and $\mathcal{F} \in D_{\mathrm{ctf}}(X, \Lambda)$.

- In the geometric case (G), how to compute $\chi_c(X_{\bar{k}}, \mathcal{F}) = \dim Rf_!\mathcal{F}$?
- In the arithmetic case (A), how to compute $\mathrm{Sw}Rf_!\mathcal{F}$, $\mathrm{dimtot}Rf_!\mathcal{F}$ and $a_S(Rf_!\mathcal{F})$?

These problems are previously studied by Abbes [Abb00], Abbes-Saito [AS07], Bloch [Blo87], Deligne [Del72, Del11], Hu [Hu15], Kato-Saito [KS04, KS08, KS12], Laumon [Lau83], Saito [Sai17, Sai18, Sai21] and Tsushima [Tsu11].

Based on the observation that the Swan conductor can be defined through the logarithmic localized intersection product, Kato and Saito [KS08, KS12] explore ramification theory using logarithmic geometry and K-theoretic localized intersection theory. Their methodology give rise to the so-called Swan class, which can be regarded as a higher-dimensional generalization of the Swan conductor. Recently, Abe [Abe21] introduces a homotopical/∞-categorical way to study ramification theory.

In [YZ22], we use a cohomological way to study Question 1.3 by introducing a cohomological class (called non-acyclicity class) supported on the non-acyclicity locus Z (Z is the smallest closed subset of X such that $X\backslash Z \to S$ is universally locally acyclic relatively to $\mathcal{F}|_{X\backslash Z}$). Using this class, we obtain a cohomological/motivic expression for the Artin conductor (cf. (3.9.1)). Thanks to this categorical description, we could be able to consider similar problems for motives (cf. [JY22]).

1.4. **Grothendieck-Ogg-Shafarevich formula.** Consider the geometric case (G). Assume X is a proper smooth connected curve over $S = \mathrm{Spec} k$. Then the Euler-Poincaré characteristic $\chi(X_{\bar{k}}, \mathcal{F})$ is computed by the Grothendieck-Ogg-Shafarevich (GOS) formula

$$\chi(X_{\bar{k}}, \mathcal{F}) = \mathrm{dim}\mathcal{F}_{\bar{\xi}} \cdot \chi(X_{\bar{k}}, \Lambda) - \sum_{x \in Z} a_x(\mathcal{F}) \cdot \deg(x), \tag{1.4.1}$$

where ξ is the generic point of X, Z is a finite set of closed points such that $\mathcal{F}|_{X \setminus Z}$ is smooth and $a_x(\mathcal{F}) = a_{X_{(x)}}(\mathcal{F})$ is the Artin conductor of $\mathcal{F}$ at x. By the Gauss-Bonnet-Chern formula $\chi(X_{\bar{k}}, \Lambda) = \deg(c_1(\Omega^{1,\vee}_{X/k}) \cap [X])$, the formula (1.4.1) can be rewritten as follows

$$\chi(X_{\bar{k}}, \mathcal{F}) = \deg(cc_{X/k}(\mathcal{F})), \tag{1.4.2}$$

where $cc_{X/k}(\mathcal{F})$ is a zero-cycle class on X:

$$cc_{X/k}(\mathcal{F}) = \mathrm{dim}\mathcal{F}_{\bar{\xi}} \cdot c_1(\Omega^{1,\vee}_{X/k}) \cap [X] - \sum_{x \in Z} a_x(\mathcal{F}) \cdot [x] \quad \text{in} \quad \mathrm{CH}_0(X). \tag{1.4.3}$$

1.5. **Characteristic cycle.** There is a generalization of the GOS formula to higher dimensional case by using characteristic cycles.[2] In the transcendental setting [KS90], Kashiwara and Schapira give a microlocal description of the characteristic cycle for a constructible sheaf $\mathcal{F}$ on a manifold without using $\mathcal{D}$-modules. In [Bei07], Beilinson asks if there is a motivic (ℓ-adic or de Rham) counterpart for their theory. As observed by Deligne, there is a strong analogy between the wild ramification of étale constructible sheaves in positive characteristic and the irregular singularity of partial differential equations on a complex manifold. In [Del11], Deligne proposes a general program to define characteristic cycles of constructible étale sheaves. Deligne's program is achieved by Saito [Sai17, Theorem 4.9 and Theorem 6.13] based on the singular support defined by Beilinson [Bei16].

Let X be a connected smooth variety of dimension n over k. For any $\mathcal{F} \in D_{\mathrm{ctf}}(X, \Lambda)$, the singular support $SS(\mathcal{F})$ is the smallest closed conical subset of the cotangent bundle T^*X such that locally on X, every function $f \colon X \to \mathbb{A}^1_k$ with df disjoint from $SS(\mathcal{F})$ is locally acyclic relatively to $\mathcal{F}$. It is proved in [Bei16] that $SS(\mathcal{F})$ is of dimension n. Later, Saito [Sai17] constructs an n-cycle $CC(\mathcal{F})$ supported on $SS(\mathcal{F})$ with $\mathbb{Z}$-coefficients, which satisfies the following properties

(a) (Index formula) Assume that X is projective[3] over a perfect field k. Then we have

$$\chi(X_{\bar{k}}, \mathcal{F}) = (CC(\mathcal{F}), T^*_X X))_{T^*X}, \tag{1.5.1}$$

where $T^*_X X$ is the zero section of T^*X.

(b) (Milnor formula) Let $f : X \to C$ be a flat morphism from a smooth scheme X to a smooth curve C over k. Assume that x_0 is an isolated characteristic point of f with respect to $SS(\mathcal{F})$ (cf. [Sai17, Definition 3.7]). Then

$$-\mathrm{dimtot}\ R\Phi_f(\mathcal{F})_{\overline{x}_0} = (CC(\mathcal{F}), df)_{T^*X, x_0}. \tag{1.5.2}$$

(c) (Conductor formula) Let X be a smooth scheme over k and Y a smooth connected curve over k with the generic point η. Let $\mathcal{F} \in D^b_c(X, \Lambda)$. Let $f : X \to Y$ be a quasi-projective morphism such that f is proper on the support of $\mathcal{F}$ and is properly $SS(\mathcal{F})$-transversal over an open dense sub-scheme $V \subseteq Y$. For each closed point $y \in Y$, the Artin conductor $a_y(Rf_*\mathcal{F}) = \chi(X_{\overline{\eta}}, \mathcal{F}) - \chi(X_{\overline{y}}, \mathcal{F}) + \mathrm{Sw}_y R\Gamma(X_{\overline{\eta}}, \mathcal{F})$ is computed by the following (geometric) conductor formula

$$-a_y(Rf_*\mathcal{F}) = (CC(\mathcal{F}), df)_{T^*X, y}. \tag{1.5.3}$$

[2] Kato and Saito [KS08, KS12] obtain another higher dimensional GOS formula and its arithmetic version by introducing Swan classes for constructible étale sheaves. See 2.4 for more details.

[3] Abe obtains an index formula for proper varieties by using ∞-categories in [Abe21].

When $\mathcal{F}$ is the constant étale sheaf Λ, then $CC(\Lambda) = (-1)^n \cdot [T^*_X X]$ and (1.5.1) is the Gauss-Bonnet-Chern formula $\chi(X_{\bar{k}}, \Lambda) = \deg(c_n(\Omega^{1,\vee}_{X/k}) \cap [X])$. The formula (1.5.2) is equivalent to the following classical Milnor formula (cf. [Del73, Conjecture 1.9, P200])

$$-\mathrm{dimtot}\ R\Phi_f(\Lambda)_{\bar{x}_0} = (-1)^n \cdot \mathrm{length}_{\mathcal{O}_{X,x}}(\mathcal{E}xt^1(\Omega^1_{X/C}, \mathcal{O}_X))_{x_0}, \tag{1.5.4}$$

and (1.5.3) gives the classical Bloch conductor formula (cf. [Blo87]).

When $\mathcal{F}$ is tame, (1.5.2) is equivalent to the logarithmic Milnor formula (cf. [Y14, 1.12]).

When X is a smooth curve, (1.5.1) is the classical Grothendieck-Ogg-Shafarevich formula (1.4.2).

When X is a smooth surface, Deligne and Laumon defined the characteristic cycle implicitly in [Lau83] under the "non-fierce" assumption.

Remark 1.6. Takeuchi [Tak19, Tak20] obtains a variant of Saito's characteristic cycle, which is related to a Milnor-type formula for the local ε-factors of vanishing cycles.

For later convenience, we introduce the following definition.

Definition 1.7 ([Sai17, Definition 6.7.2])**.** Let $0_X : X \to T^*X$ be the zero section. The (geometric) characteristic class of $\mathcal{F}$ is defined to be the following zero class

$$cc_{X/k}(\mathcal{F}) = 0^!_X(CC(\mathcal{F})) \quad \text{in} \quad \mathrm{CH}_0(X). \tag{1.7.1}$$

1.8. **Regular case.** Consider the arithmetic case (A). We assume that S is of mixed characteristic. We still have Deligne's conjecture on the Milnor formula and Bloch's conductor formula, which are still open in general. In [Org03, Théorème 0.8], Orgogozo shows that the conductor formula of Bloch implies the Milnor formula. In [KS04, Corollary 6.2.7], Kato and Saito show that the conductor formula is a consequence of an embedded resolution of singularities in a strong sense for the reduced closed fiber. Hence, the Milnor formula is true if we assume an embedded resolution. In particular, the Milnor formula is true if the relative dimension is equal to one or two.

It is nature to ask whether there is an arithmetic version for the index formula (1.5.1). In [Abb00], Abbes proves an arithmetic Grothendieck-Ogg-Shafarevich formula for curves over a local field. A higher general version is obtained by Kato and Saito in [KS04]. In [Sai22], Saito defines the FW-cotangent bundle for a regular scheme. By using cohomological transversality conditions, he also introduces the singular support for a constructible complex on a regular scheme (without proving the existence). It is interesting to see whether the Milnor formula and Bloch's conductor formula in the mixed characteristic case could be reformulated in terms of FW-cotangent bundles. It is possible for arithmetic surfaces [Ooe24]. We will study this question through cohomological methods in future (See "Research project I") by constructing an arithmetic version of the non-acyclicity class.

2. Characteristic classes and epsilon factors

2.1. **Epsilon factors.** The L-function and the ε-factor are two important objects in geometric Langlands program and Galois representation theory. Ramification theory will appear behind these objects in a natural way.

Let k be a finite field of characteristic p and let $f\colon X \to \mathrm{Spec}(k)$ be a smooth projective morphism purely of dimension d. Let Λ be a finite field of characteristic $\ell \neq p$. For a constructible complex $\mathcal{F}$ of Λ-modules on X, let $D(\mathcal{F})$ be the dual $R\mathcal{H}om(\mathcal{F}, \mathcal{K}_X)$ of $\mathcal{F}$ where $\mathcal{K}_X = Rf^!\Lambda$ is the dualizing complex. The L-function $L(X, \mathcal{F}, t)$ satisfies the following functional equation

$$L(X, \mathcal{F}, t) = \varepsilon(X, \mathcal{F}) \cdot t^{-\chi(X_{\bar{k}}, \mathcal{F})} \cdot L(X, D(\mathcal{F}), t^{-1}), \tag{2.1.1}$$

where

$$\varepsilon(X, \mathcal{F}) = \det(-\mathrm{Frob}_k; R\Gamma(X_{\bar{k}}, \mathcal{F}))^{-1} \tag{2.1.2}$$

is the epsilon factor (the constant term in the functional equation (2.1.1)) and $\chi(X_{\bar{k}},\mathcal{F})$ is the Euler-Poincaré characteristic of $\mathcal{F}$. In the functional equation (2.1.1), both $\chi(X_{\bar{k}},\mathcal{F})$ and $\varepsilon(X,\mathcal{F})$ are related to the ramification theory. Indeed, $\chi(X_{\bar{k}},\mathcal{F})=\deg cc_X(\mathcal{F})$ (cf. (1.5.1) and (1.7.1)). For the epsilon factor, it is more complicated.

2.2. **Twist formula.** Let $\rho_X\colon CH_0(X)\to\pi_1(X)^{\mathrm{ab}}$ be the reciprocity map which is defined by sending the class $[s]$ of a closed point $s\in X$ to the geometric Frobenius Frob_s. Let $\mathcal{G}$ be a smooth sheaf on X and $\det\mathcal{G}:\pi_1(X)^{\mathrm{ab}}\to\Lambda^\times$ be the character associated to the determinant sheaf $\det\mathcal{G}$. In joint work with Umezaki and Zhao [UYZ20], we prove the following twist formula:

$$\varepsilon(X,\mathcal{F}\otimes\mathcal{G})=\det\mathcal{G}(-cc_X(\mathcal{F}))\cdot\varepsilon(X,\mathcal{F})^{\mathrm{rank}\mathcal{G}},\tag{2.2.1}$$

which is conjectured by Kato and Saito in [KS08, Conjecture 4.3.11].[4] When $\mathcal{F}$ is the constant sheaf Λ, this is proved in [Sai84]. If $\mathcal{F}$ is a smooth sheaf on an open dense subscheme U of X such that the complement $D=X\backslash U$ is a simple normal crossing divisor and the sheaf $\mathcal{F}$ is tamely ramified along D, then (2.2.1) is a consequence of [Sai93, Theorem 1]. If $\dim(X)=1$, the formula (2.2.1) follows from the product formula of Deligne and Laumon (cf. [Del72e, 7.11] and [Lau87, 3.2.1.1]). In [Vid09a, Vid09b], Vidal proves a similar result on a proper smooth surface over a finite field of characteristic $p>2$ under some technical assumptions.

As a corollary of (2.2.1), we prove the compatibility of the characteristic class with proper push-forward by using the injectivity of the reciprocity map ρ_X [KS83, Theorem 1]. In general, Saito proves that the characteristic cycle (resp. characteristic class) is compatible with proper push-forward under a mild assumption (cf. [Sai17, 7.2], [Sai18, Conjecture 1] and [Sai21, Theorem 2.2.5]). In [YZ21], we also prove a relative version of the twist formula (2.5.1).

Question 2.3. Prove a similar formula of (2.2.1) if $\mathcal{G}$ is smooth only on an open dense subscheme $U\subseteq X$ such that its wild ramification along $X\backslash U$ is much smaller than that of $\mathcal{F}$.

2.4. **Swan class.** To generalize the classical Grothendieck-Ogg-Shafarevich formula for curves to higher dimensional varieties, Kato and Saito define the so-called Swan class in [KS08]. Saito formulates a conjecture that this object should be re-defined using the characteristic cycle (cf. [Sai17, Conjecture 5.8]). More precisely, let X be a smooth scheme over a perfect field k of characteristic $p>0$, and $\overline{X}$ a smooth compactification of X. For a smooth and constructible sheaf $\mathcal{F}$ of Λ-modules on X, Saito conjectures that the Swan class of $\mathcal{F}$ should have integer coefficients and is equal to the pull back by the zero section of the difference $CC(j_!\mathcal{F})-\mathrm{rank}\mathcal{F}\cdot CC(j_!\Lambda)$. In [UYZ20], we verify a weaker version of this conjecture for smooth surfaces over a finite field. Our method also works for higher dimensional varieties if we assume resolution of singularities and a special case of proper push-forward of characteristic class (cf. [UYZ20, Theorem 6.6]).

2.5. **Relative twist formula.** In [YZ21, 2.1], we formulate a relative version of Kato-Saito's formula and prove it under certain transversality conditions. Let S be a regular Noetherian scheme over $\mathbb{Z}[1/\ell]$ and $f:X\to S$ a smooth proper morphism purely of relative dimension n. Let Λ be a finite field of characteristic ℓ or $\Lambda=\overline{\mathbb{Q}}_\ell$. Let $\mathcal{F}\in D_c^b(X,\Lambda)$ such that f is universally locally acyclic relatively to $\mathcal{F}$. We conjecture that there is a (relative) cycle class $cc_{X/S}(\mathcal{F})\in\mathrm{CH}^n(X)$ such that for any smooth sheaf $\mathcal{G}$ of Λ-modules on X, we have an isomorphism

$$\det Rf_*(\mathcal{F}\otimes^L\mathcal{G})\simeq(\det Rf_*\mathcal{F})^{\otimes\mathrm{rank}\mathcal{G}}\otimes^L\det\mathcal{G}(cc_{X/S}(\mathcal{F}))\quad\text{in}\quad K_0(S,\Lambda),\tag{2.5.1}$$

where $K_0(S,\Lambda)$ is the Grothendieck group of $D_c^b(S,\Lambda)$. In (2.5.1), the object $\det\mathcal{G}(cc_{X/S}(\mathcal{F}))$ is a smooth sheaf of rank 1 determined as follows:

$$\pi_1^{\mathrm{ab}}(S)\xrightarrow{(cc_{X/S}(\mathcal{F}),-)}\pi_1^{\mathrm{ab}}(X)\xrightarrow{\det\mathcal{G}}\Lambda^\times,\tag{2.5.2}$$

[4] The original conjecture is formulated in terms of the Swan class.

where the pairing is given by $\mathrm{CH}^n(X) \times \pi_1^{\mathrm{ab}}(S) \to \pi_1^{\mathrm{ab}}(X)$ (cf. [Sai94, Proposition 1]).

When S is a smooth scheme over a perfect field k, we construct a candidate for $cc_{X/S}(\mathcal{F})$ in [YZ21, Definition 2.11] by using the characteristic cycle of $CC(\mathcal{F})$. As an evidence, we prove a special case of the conjectural formula (2.5.1) in [YZ21, Theorem 2.12].

From the above relative twist formula, we realize that there is a relative version of the cohomological characteristic class (cf. [YZ21, Definition 3.6]) under certain transversality conditions. We also prove a relative Lefschetz-Verdier trace formula in [YZ21, Theorem 3.9]. These results are further generalized to ULA sheaves by Lu and Zheng [LZ22] by using categorical traces.

2.6. **Microlocal description.** Let R be a commutative ring. Let $\mathcal{F}$ be a perfect constructible complex of sheaves of R-modules on a compact real analytic manifold X. In [Bei07], Beilinson develops the theory of topological epsilon factors using K-theory spectrum. More precisely, he gives a Dubson-Kashiwara-style description of $\det R\Gamma(X, \mathcal{F})$, and he asks that whether the construction admits a motivic (ℓ-adic or de Rham) counterpart. For de Rham cohomology, such a construction is given by his PhD student Patel in [Pat12]. Based on these, Abe and Patel prove a similar twist formula in [AP18] for global de Rham epsilon factors in the classical setting of $\mathcal{D}_X$-modules on smooth projective varieties over a field of characteristic zero. As pointed out by Abe and Patel, proving the formula at the level of K-theory spectra should also give formulas in higher K-theory. At the level of K_0 (resp. K_1), one gets formulas for the Euler characteristic (resp. determinants). It would be interesting to see the consequences at the level of K_2 (or higher K-groups).

For ℓ-adic cohomology, Beilinson's question is still open. For a constructible étale sheaf $\mathcal{F}$ on a smooth curve X over a finite field k, the precise statement for the ε-factorization of

$$\det(-\mathrm{Frob}_k; R\Gamma(X, \mathcal{F}))$$

was conjectured by Deligne [Del72e] and proved by Laumon [Lau87] using local Fourier transform and ℓ-adic version of principle of stationary phase. A higher dimensional analogue is obtained by Guignard [Gui22] (see also [Tak19]).

2.7. **Citation.** Our work [UYZ20] on Kato-Saito's conjecture is cited by [AP18, Sai21, Gui22, YZ21, YZ22] and also by the following papers:

(1) W. Sawin, A. Forey, J. Fresán and E. Kowalski, *Quantitative sheaf theory*, Journal of the American Mathematical Society, 36(3), (2023): 653-726.
(2) D. Patel and K. V. Shuddhodan, *Brylinski-Radon transformation in characteristic* $p > 0$, preprint arXiv:2307.04156, 2023.
(3) D. Takeuchi, *Characteristic epsilon cycles of ℓ-adic sheaves on varieties*, arXiv:1911.02269, 2019.
(4) F. Orgogozo and J. Riou, *Cycle caractéristique sur une puissance symétrique d'une courbe et déterminant de la cohomologie étale*, arXiv:2312.07776, 2023.
(5) A. Rai, *Comparison of the two notions of characteristic cycles*, arXiv:2312.09945, 2023.

3. Non-acyclicity class and Saito's conjecture

3.1. Let $h : X \to \mathrm{Spec}k$ be a separated morphism of finite type over a perfect field k. Let $\mathcal{K}_{X/k} = Rh^!\Lambda$. For any object $\mathcal{F} \in D_{\mathrm{ctf}}(X, \Lambda)$, the cohomological characteristic class $C_{X/k}(\mathcal{F}) \in H^0(X, \mathcal{K}_{X/k})$ is introduced by Abbes and Saito in [AS07] by using Verdier pairing. If X is proper over k, the Lefschetz-Verdier trace formula gives

$$\chi(X_{\bar{k}}, \mathcal{F}) = \mathrm{Tr}C_{X/k}(\mathcal{F}), \tag{3.1.1}$$

where $\mathrm{Tr} : H^0(X, \mathcal{K}_{X/k}) \to \Lambda$ is the trace map.

Using ramification theory, Abbes and Saito calculate the cohomological characteristic classes for rank 1 sheaves under certain ramification conditions in [AS07]. However, the calculation for general constructible étale sheaves remains an outstanding question in ramification theory. In general, Saito proposes the following conjecture.

Conjecture 3.2 (Saito, [Sai17, Conjecture 6.8.1]). *Let X be a closed sub-scheme of a smooth scheme over a perfect field k. Let $\mathcal{F}$ be a constructible complex of Λ-modules of finite tor-dimension on X. Consider the characteristic class $cc_{X/k}(\mathcal{F})$ defined by (1.7.1). Then we have*

$$C_{X/k}(\mathcal{F}) = \mathrm{cl}(cc_{X/k}(\mathcal{F})) \quad \text{in} \quad H^0(X, \mathcal{K}_{X/k}), \tag{3.2.1}$$

where $\mathrm{cl} : \mathrm{CH}_0(X) \to H^0(X, \mathcal{K}_{X/k})$ *is the cycle class map.*

Note that when X is projective and smooth over a finite field k of characteristic p, the cohomology group $H^0(X, \mathcal{K}_{X/k})$ is highly non-trivial. For example, if $\Lambda = \mathbb{Z}/\ell^m$ with $\ell \neq p$, then we have $H^0(X, \mathcal{K}_{X/k}) \simeq H^1(X, \mathbb{Z}/\ell^m)^\vee \simeq \pi_1^{\mathrm{ab}}(X)/\ell^m$.

Saito's conjecture says that the cohomological characteristic class can be computed in terms of the characteristic cycle. Note that the two involved ramification invariants in Conjecture 3.2 are defined in quite different ways. The characteristic cycle is characterized by the Milnor formula (1.5.2), while the cohomological characteristic class in some sense is defined via the categorical trace. In the characteristic zero case, the equality (3.2.1) on a complex manifold is the microlocal index formula proved by Kashiwara and Schapira [KS90, 9.5.1]. However, we don't know such a microlocal description for characteristic cycles in positive characteristic (but see [AS09, Abe21]). In [YZ22], we prove the quasi-projective case of Saito's conjecture.

Theorem 3.3 ([YZ22, Theorem 1.3]). *Conjecture 3.2 holds for any smooth and quasi-projective scheme X over a perfect field k of characteristic $p > 0$.*

3.4. Our approach to Saito's conjecture is the fibration method, which leans on the construction of the relative version of cohomological characteristic classes over a general base scheme. To achieve this, it is necessary to impose additional transversality conditions on the structure morphism. Let S be a Noetherian scheme. Let $h : X \to S$ be a separated morphism of finite type, $\mathcal{K}_{X/S} = Rh^!\Lambda$ and $\mathcal{F} \in D_{\mathrm{ctf}}(X, \Lambda)$. In fact, under certain smooth and transversality conditions on h, we introduce the relative (cohomological) characteristic class $C_{X/S}(\mathcal{F}) \in H^0(X, \mathcal{K}_{X/S})$ in [YZ21, Definition 3.6]. It is further generalized to any separated morphism $h : X \to S$ which is (universally) locally acyclic relatively to $\mathcal{F}$ by using categorical traces [LZ22, 2.20]. We also define relative characteristic classes in a more general case that only assumes local acyclicity away from small closed subsets. Indeed, if $Z \subseteq X$ is a closed subscheme such that $H^0(Z, \mathcal{K}_{Z/S}) = H^1(Z, \mathcal{K}_{Z/S}) = 0$, and if $X \backslash Z \to S$ is (universally) locally acyclic relatively to $\mathcal{F}|_{X\backslash Z}$, then the relative characteristic class $C_{X/S}(\mathcal{F}) \in H^0(X, \mathcal{K}_{X/S})$ remains well-defined (cf. [YZ22, Definition 3.5]). We prove the following fibration formula for cohomological characteristic classes.

Theorem 3.5 ([YZ22, Theorem 6.5]). *Let Y be a smooth connected curve over a perfect field k of characteristic $p > 0$. Let Λ be a finite local ring such that the characteristic of the residue field is invertible in k. Let $f : X \to Y$ be a separated morphism of finite type and $Z \subseteq |X|$ be a finite set of closed points. Let $\mathcal{F} \in D_{\mathrm{ctf}}(X, \Lambda)$ such that $f|_{X\backslash Z}$ is universally locally acyclic relatively to $\mathcal{F}|_{X\backslash Z}$. Then we have*

$$C_{X/k}(\mathcal{F}) - c_1(f^*\Omega^{1,\vee}_{Y/k}) \cap C_{X/Y}(\mathcal{F}) = -\sum_{x \in Z} \mathrm{dimtot} R\Phi_f(\mathcal{F})_{\bar{x}} \cdot [x] \quad \text{in} \quad H^0(X, \mathcal{K}_{X/k}). \tag{3.5.1}$$

In [UYZ20, Proposition 5.3.7], the characteristic class $cc_{X/k}(\mathcal{F})$ satisfies a similar fibration formula. The formula (3.5.1) implies that the characteristic class $C_{X/k}(\mathcal{F})$ can be built from characteristic classes $C_{X_v/v}(\mathcal{F}|_{X_v})$ $(v \in |Y\backslash Z|)$ on schemes X_v of dimension smaller than X. Thus we can prove Saito's conjecture by induction on the dimension of X, and the curve case follows from the Grothendieck-Ogg-Shafarevich formula (1.4.3) and its cohomological version (curve case of (3.5.1)).

3.6. In order to prove Theorem 3.5, we have to give a purely cohomological/categorical way to define the right hand side of (3.5.1), i.e., we have to define a cohomological class supported on the non-acyclicity locus. Let S be a Noetherian scheme. Consider a commutative diagram in Sch_S:

$$\begin{array}{ccccc} Z & \overset{\tau}{\hookrightarrow} & X & \xrightarrow{f} & Y, \\ & & {\scriptstyle h}\searrow & & \swarrow{\scriptstyle g} \\ & & & S & \end{array} \tag{3.6.1}$$

where $\tau : Z \to X$ is a closed immersion and g is a smooth morphism. We define an object $\mathcal{K}_{X/Y/S}$ on X sitting in a distinguished triangle (see also [YZ22, (4.2.5)])

$$\mathcal{K}_{X/Y} \to \mathcal{K}_{X/S} \to \mathcal{K}_{X/Y/S} \xrightarrow{+1} . \tag{3.6.2}$$

Let $\mathcal{F} \in D_{\mathrm{ctf}}(X, \Lambda)$ such that $X \backslash Z \to Y$ is universally locally acyclic relatively to $\mathcal{F}|_{X\backslash Z}$ and that $h : X \to S$ is universally locally acyclic relatively to $\mathcal{F}$. In [YZ22, Definition 4.6], we introduce the non-acyclicity class $\widetilde{C}^Z_{X/Y/S}(\mathcal{F}) \in H^0_Z(X, \mathcal{K}_{X/Y/S})$ supported on Z. If the following condition holds:

$$H^0(Z, \mathcal{K}_{Z/Y}) = 0 \text{ and } H^1(Z, \mathcal{K}_{Z/Y}) = 0 \tag{3.6.3}$$

then the map $H^0_Z(X, \mathcal{K}_{X/S}) \xrightarrow{(3.6.2)} H^0_Z(X, \mathcal{K}_{X/Y/S})$ is an isomorphism. In this case, the class $\widetilde{C}^Z_{X/Y/S}(\mathcal{F}) \in H^0_Z(X, \mathcal{K}_{X/Y/S})$ defines an element of $H^0_Z(X, \mathcal{K}_{X/S})$, which is denoted by $C^Z_{X/Y/S}(\mathcal{F})$.

In the case that $X = Y \to S = \mathrm{Spec} k$ is smooth over a field k. Since $\mathrm{id} : X\backslash Z \to X\backslash Z$ is universally locally acyclic relatively to $\mathcal{F}|_{X\backslash Z}$, the cohomology sheaves of $\mathcal{F}|_{X\backslash Z}$ are locally constant on $X\backslash Z$. In this case, the class $C^Z_{X/Y/S}(\mathcal{F})$ is Abbes-Saito's localized characteristic class [AS07, Definition 5.2.1].

Now we summarize the functorial properties for the non-acyclicity classes.

Theorem 3.7 ([YZ22, Theorem 1.9, Proposition 1.11, Theorem 1.12, Theorem 1.14]). *Let us denote the diagram (3.6.1) simply by $\Delta = \Delta^Z_{X/Y/S}$ and $\widetilde{C}^Z_{X/Y/S}(\mathcal{F})$ by $C_\Delta(\mathcal{F})$. Let $\mathcal{F} \in D_{\mathrm{ctf}}(X, \Lambda)$. Assume that $Y \to S$ is smooth, $X\backslash Z \to Y$ is universally locally acyclic relatively to $\mathcal{F}|_{X\backslash Z}$ and that $X \to S$ is universally locally acyclic relatively to $\mathcal{F}$.*

(1) (*Fibration formula*) *If $H^0(Z, \mathcal{K}_{Z/Y}) = H^1(Z, \mathcal{K}_{Z/Y}) = 0$, then we have*

$$C_{X/S}(\mathcal{F}) = c_r(f^*\Omega^{1,\vee}_{Y/S}) \cap C_{X/Y}(\mathcal{F}) + C_\Delta(\mathcal{F}) \quad \text{in} \quad H^0(X, \mathcal{K}_{X/S}). \tag{3.7.1}$$

(2) (*Pull-back*) *Let $b : S' \to S$ be a morphism of Noetherian schemes. Let $\Delta' = \Delta^{Z'}_{X'/Y'/S'}$ be the base change of $\Delta = \Delta^Z_{X/Y/S}$ by $b : S' \to S$. Let $b_X : X' = X \times_S S' \to X$ be the base change of b by $X \to S$. Then we have*

$$b_X^* C_\Delta(\mathcal{F}) = C_{\Delta'}(b_X^*\mathcal{F}) \quad \text{in} \quad H^0_{Z'}(X', \mathcal{K}_{X'/Y'/S'}), \tag{3.7.2}$$

where $b_X^* : H^0_Z(X, \mathcal{K}_{X/Y/S}) \to H^0_{Z'}(X', \mathcal{K}_{X'/Y'/S'})$ is the induced pull-back morphism.

(3) (*Proper push-forward*) *Consider a diagram $\Delta' = \Delta^{Z'}_{X'/Y/S}$. Let $s : X \to X'$ be a proper morphism over Y such that $Z \subseteq s^{-1}(Z')$. Then we have*

$$s_*(C_\Delta(\mathcal{F})) = C_{\Delta'}(Rs_*\mathcal{F}) \quad \text{in} \quad H^0_{Z'}(X', \mathcal{K}_{X'/Y/S}), \tag{3.7.3}$$

where $s_* : H^0_Z(X, \mathcal{K}_{X/Y/S}) \to H^0_{Z'}(X', \mathcal{K}_{X'/Y/S})$ is the induced push-forward morphism.

(4) (*Cohomological Milnor formula*) *Assume* $S = \operatorname{Spec} k$ *for a perfect field* k *of characteristic* $p > 0$ *and* Λ *is a finite local ring such that the characteristic of the residue field is invertible in* k. *If* $Z = \{x\}$, *then we have*

$$C_{\Delta}(\mathcal{F}) = -\mathrm{dimtot} R\Phi_f(\mathcal{F})_{\bar{x}} \quad \text{in} \quad \Lambda = H^0_x(X, \mathcal{K}_{X/k}). \tag{3.7.4}$$

(5) (*Cohomological conductor formula*) *Assume* $S = \operatorname{Spec} k$ *for a perfect field* k *of characteristic* $p > 0$ *and* Λ *is a finite local ring such that the characteristic of the residue field is invertible in* k. *If* Y *is a smooth connected curve over* k *and* $Z = f^{-1}(y)$ *for a closed point* $y \in |Y|$, *then we have*

$$f_* C_{\Delta}(\mathcal{F}) = -a_y(Rf_*\mathcal{F}) \quad \text{in} \quad \Lambda = H^0_y(Y, \mathcal{K}_{Y/k}). \tag{3.7.5}$$

The formation of non-acyclicity classes is also compatible with specialization maps (cf. [YZ22, Proposition 4.17]).

Now Theorem (3.5) follows from (3.7.1) and (3.7.4). By verifying certain diagrams commute, one could prove (3.7.1)-(3.7.3). The proof of (3.7.4) is based on (3.7.2) together with a homotopy argument in [Abe22]. The formula (3.7.5) follows from (3.7.3) and (3.7.4). To prove (3.7.1), we enhance the constructions of $C_{X/S}$ and $C^Z_{X/Y/S}$ to the ∞-categorical level and construct an intermediate map $L^Z_{X/Y/S}(\mathcal{F})$ together with a coherent commutative diagram

$$\begin{array}{ccccc} & & \Lambda & = & \Lambda \\ & \swarrow^{C^Z_{X/Y/S}(\mathcal{F})} & \downarrow{\scriptstyle L^Z_{X/Y/S}(\mathcal{F})} & & \downarrow{\scriptstyle C_{X/S}(\mathcal{F}) - \delta^! C_{X/Y}(\mathcal{F})} \\ \tau_*\tau^!\mathcal{K}_{X/Y/S} & \longleftarrow & \tau_*\mathcal{K}_{Z/S} & \longrightarrow & \mathcal{K}_{X/S}. \end{array} \tag{3.7.6}$$

Since $H^0_Z(X, \mathcal{K}_{X/Y/S}) \simeq H^0(Z, \mathcal{K}_{Z/S})$, the diagram (3.7.6) implies the fibration formula (3.7.1).

Remark 3.8. If we apply the non-acyclicity class to the following diagram constructed by Saito in [Sai17, p.652, (5.13)]

$$\begin{array}{ccccc} \mathbf{Z}(\widetilde{C}) & \hookrightarrow & (X \times \mathbf{G})^{\triangledown} & \longrightarrow & \mathbf{D}, \\ & & & \searrow \quad \swarrow & \\ & & & \mathbf{G} & \end{array} \tag{3.8.1}$$

we could be able to recover the characteristic cycle $CC(\mathcal{F})$ in a weaker sense.

3.9. **Cohomological expression for Artin conductors.** Let X be a smooth connected curve over k. Let $\mathcal{F} \in D_{\mathrm{ctf}}(X, \Lambda)$ and $Z \subseteq X$ be a finite set of closed points such that the cohomology sheaves of $\mathcal{F}|_{X \setminus Z}$ are locally constant. By the cohomological Milnor formula (3.7.4), we have the following (motivic) expression for the Artin conductor of $\mathcal{F}$ at $x \in Z$

$$a_x(\mathcal{F}) = \mathrm{dimtot} R\Phi_{\bar{x}}(\mathcal{F}, \mathrm{id}) = -C^{\{x\}}_{U/U/k}(\mathcal{F}|_U), \tag{3.9.1}$$

where U is any open subscheme of X such that $U \cap Z = \{x\}$. By (3.7.1), we get the following cohomological Grothendieck-Ogg-Shafarevich formula (cf. [YZ22, Corollary 6.6]):

$$C_{X/k}(\mathcal{F}) = \mathrm{rank}\mathcal{F} \cdot c_1(\Omega^{1,\vee}_{X/k}) - \sum_{x \in Z} a_x(\mathcal{F}) \cdot [x] \quad \text{in} \quad H^0(X, \mathcal{K}_{X/k}). \tag{3.9.2}$$

Based on the observation (3.9.1), we could be able to study the ramification theory for motives and get a quadratic version of the GOS formula (cf. [JY22]).

3.10. **Milnor-type formula for non-isolated singularities.** In [XY23], we construct the geometric counterpart of the non-acyclicity class and propose a Milnor-type formula for non-isolated singularities. The conjecture says that the non-acyclicity classes can be calculated in terms of the characteristic cycles.

4. Transversality condition

In this section, we recall the definition of the non-acyclicity class. To simplify our notation, we omit to write R or L to denote the derived functors unless otherwise stated explicitly or for $R\mathcal{H}om$.

4.1. **Transversality condition.** We recall the (cohomological) transversality condition introduced in [YZ22, 2.1], which is a relative version of the transversality condition studied by Saito [Sai17, Definition 8.5]. Let S be a Noetherian scheme and Λ a Noetherian ring such that $m\Lambda = 0$ for some integer m invertible on S. Consider the following cartesian diagram in Sch_S:

(4.1.1)
$$\begin{array}{ccc} X & \xrightarrow{i} & Y \\ {\scriptstyle p}\downarrow & \square & \downarrow{\scriptstyle f} \\ W & \xrightarrow{\delta} & T. \end{array}$$

Let $\mathcal{F} \in D_{\mathrm{ctf}}(Y, \Lambda)$ and $\mathcal{G} \in D_{\mathrm{ctf}}(T, \Lambda)$. Let $c_{\delta,f,\mathcal{F},\mathcal{G}}$ be the composition

(4.1.2)
$$\begin{aligned} c_{\delta,f,\mathcal{F},\mathcal{G}} : i^*\mathcal{F} \otimes^L p^*\delta^!\mathcal{G} &\xrightarrow{id \otimes \mathrm{b.c}} i^*\mathcal{F} \otimes^L i^! f^*\mathcal{G} \xrightarrow{\mathrm{adj}} i^! i_!(i^*\mathcal{F} \otimes^L i^! f^*\mathcal{G}) \\ &\xrightarrow[\simeq]{\mathrm{proj.formula}} i^!(\mathcal{F} \otimes^L i_! i^! f^*\mathcal{G}) \xrightarrow{\mathrm{adj}} i^!(\mathcal{F} \otimes^L f^*\mathcal{G}). \end{aligned}$$

We put $c_{\delta,f,\mathcal{F}} := c_{\delta,f,\mathcal{F},\Lambda} : i^*\mathcal{F} \otimes^L p^*\delta^!\Lambda \to i^!\mathcal{F}$. If $c_{\delta,f,\mathcal{F}}$ is an isomorphism, then we say that the morphism δ is $\mathcal{F}$*-transversal*. If $c_{i,\mathrm{id},\mathcal{F}}$ is an isomorphism, then we say i is $\mathcal{F}$-transversal.

By [YZ22, 2.11], there is a functor $\delta^\Delta : D_{\mathrm{ctf}}(Y, \Lambda) \to D_{\mathrm{ctf}}(X, \Lambda)$ such that for any $\mathcal{F} \in D_{\mathrm{ctf}}(Y, \Lambda)$, we have a distinguished triangle

(4.1.3)
$$i^*\mathcal{F} \otimes^L p^*\delta^!\Lambda \xrightarrow{c_{\delta,f,\mathcal{F}}} i^!\mathcal{F} \to \delta^\Delta\mathcal{F} \xrightarrow{+1}.$$

Then δ is $\mathcal{F}$-transversal if and only if $\delta^\Delta(\mathcal{F})$=0 (cf. [YZ22, Lemma 2.12]). If δ is a closed immersion and $j : T\backslash W \to T$ is the open immersion, then we have

(4.1.4)
$$\delta^\Delta\mathcal{F} := i^!(\mathcal{F} \otimes^L f^* j_*\Lambda).$$

The following lemma gives an equivalence characterization between transversality condition and (universal) local acyclicity condition (cf. [XY23, Lemma 2.2]).

Lemma 4.2. *Let $f : X \to S$ be a morphism of finite type between Noetherian schemes and $\mathcal{F} \in D_{\mathrm{ctf}}(X, \Lambda)$. The following conditions are equivalent:*

1. *The morphism f is (universally) locally acyclic relatively to $\mathcal{F}$.*
2. *For any $\mathcal{G} \in D_{\mathrm{ctf}}(X, \Lambda)$, the canonical map*

(4.2.1)
$$D_{X/S}(\mathcal{G}) \boxtimes^L \mathcal{F} \to R\mathcal{H}om_{X\times_S X}(\mathrm{pr}_1^*\mathcal{G}, \mathrm{pr}_2^!\mathcal{F})$$

is an isomorphism in $D_{\mathrm{ctf}}(X \times_S X, \Lambda)$, where $\mathrm{pr}_1 : X \times_S X \to X$ and $\mathrm{pr}_2 : X \times_S X \to X$ are the projections, $D_{X/S}(\mathcal{F}) = R\mathcal{H}om(\mathcal{G}, \mathcal{K}_{X/S})$ and $\mathcal{K}_{X/S} = Rf^!\Lambda$.

(3) *For any cartesian diagram between Noetherian schemes*

$$\begin{array}{ccc} Y\times_S X & \xrightarrow{\mathrm{pr}_2} & X \\ {\scriptstyle \mathrm{pr}_1}\downarrow & \square & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[\delta]{} & S \end{array} \tag{4.2.2}$$

and any $\mathcal{G}\in D_{\mathrm{ctf}}(S,\Lambda)$, the morphism $c_{\delta,f,\mathcal{F},\mathcal{G}}$ is an isomorphism (in particular, δ is $\mathcal{F}$-transversal).

4.3. **Non-acyclicity class.** Consider the commutative diagram (3.6.1). Let $i: X\times_Y X\to X\times_S X$ be the base change of the diagonal morphism $\delta: Y\to Y\times_S Y$:

$$\begin{array}{ccc} X & = & X \\ {\scriptstyle \delta_1}\downarrow & \square & \downarrow{\scriptstyle \delta_0} \\ X\times_Y X & \xrightarrow{i} & X\times_S X \\ {\scriptstyle p}\downarrow & \square & \downarrow{\scriptstyle f\times f} \\ Y & \xrightarrow{\delta} & Y\times_S Y, \end{array} \qquad (f: X\to Y) \tag{4.3.1}$$

where δ_0 and δ_1 are the diagonal morphisms. Put $\mathcal{K}_{X/Y/S}:=\delta^{\Delta}\mathcal{K}_{X/S}\simeq\delta_1^*\delta^{\Delta}\delta_{0*}\mathcal{K}_{X/S}$. By (4.1.3), we have the following distinguished triangle (cf. [YZ22, (4.2.5)])

$$\mathcal{K}_{X/Y}\to\mathcal{K}_{X/S}\to\mathcal{K}_{X/Y/S}\xrightarrow{+1}. \tag{4.3.2}$$

Let $\mathcal{F}\in D_{\mathrm{ctf}}(X,\Lambda)$ such that $X\backslash Z\to Y$ is universally locally acyclic relatively to $\mathcal{F}|_{X\backslash Z}$ and that $h: X\to S$ is universally locally acyclic relatively to $\mathcal{F}$. We put

$$\mathcal{H}_S=R\mathcal{H}om_{X\times_S X}(\mathrm{pr}_2^*\mathcal{F},\mathrm{pr}_1^!\mathcal{F}),\qquad \mathcal{T}_S=\mathcal{F}\boxtimes_S^L D_{X/S}(\mathcal{F}). \tag{4.3.3}$$

The relative cohomological characteristic class $C_{X/S}(\mathcal{F})$ is the composition (cf. [YZ22, 3.1])

$$\Lambda\xrightarrow{\mathrm{id}}R\mathcal{H}om(\mathcal{F},\mathcal{F})\simeq\delta_0^!\mathcal{H}_S\xleftarrow[\simeq]{(4.2.1)}\delta_0^!\mathcal{T}_S\to\delta_0^*\mathcal{T}_S\xrightarrow{\mathrm{ev}}\mathcal{K}_{X/S}. \tag{4.3.4}$$

By the assumption on $\mathcal{F}$, $\delta_1^*\delta^{\Delta}\mathcal{T}_S$ is supported on Z by [YZ22, 4.4]. The non-acyclicity class $\widetilde{C}^Z_{X/Y/S}(\mathcal{F})$ is the composition (cf. [YZ22, Definition 4.6])

$$\Lambda\to\delta_0^!\mathcal{H}_S\xleftarrow{\simeq}\delta_0^!\mathcal{T}_S\simeq\delta_1^!i^!\mathcal{T}_S\to\delta_1^*i^!\mathcal{T}_S\to\delta_1^*\delta^{\Delta}\mathcal{T}_S\xleftarrow{\simeq}\tau_*\tau^!\delta_1^*\delta^{\Delta}\mathcal{T}_S\to\tau_*\tau^!\mathcal{K}_{X/Y/S}. \tag{4.3.5}$$

If the following condition holds:

$$H^0(Z,\mathcal{K}_{Z/Y})=0\text{ and }H^1(Z,\mathcal{K}_{Z/Y})=0, \tag{4.3.6}$$

then the map $H^0_Z(X,\mathcal{K}_{X/S})\xrightarrow{(3.6.2)}H^0_Z(X,\mathcal{K}_{X/Y/S})$ is an isomorphism. In this case, the class $\widetilde{C}^Z_{X/Y/S}(\mathcal{F})\in H^0_Z(X,\mathcal{K}_{X/Y/S})$ defines an element of $H^0_Z(X,\mathcal{K}_{X/S})$, which is denoted by $C^Z_{X/Y/S}(\mathcal{F})$.

5. Ramification theory for motives

5.1. **Quadratic Artin conductor.** When I was doing postdoc at Regensburg University, Professor Denis-Charles Cisinski proposed a project on constructing the characteristic cycles for motives. To carry out this project, we have to consider the following things:

(1) Define the singular support for a constructible motivic spectrum $\mathcal{F}\in\mathbf{SH}_c(X)$ on a smooth variety X over a perfect field k.

(2) Construct a quadratic refinement of the Artin conductor in the case that X is a smooth curve. More precisely, for each closed point $x \in |X|$, we need a quadratic form $a_x^Q(\mathcal{F})$ in the Grothendieck-Witt ring $\mathrm{GW}(k(x))$ of (virtual) non-degenerate symmetric bi-linear forms over $k(x)$ such that the rank of $a_x^Q(\mathcal{F})$ equals the classical Artin conductor at x of the étale realization of $\mathcal{F}$.
(3) Formulate a quadratic refinement for the Milnor formula (1.5.2) and the conductor formula (1.5.3).
(4) Construct a quadratic version of the characteristic cycle for a nice **SH** motive.

In an ongoing note with Cisinski, we could be able to define the singular support for constructible motives following Beilinson's argument by using $\mathcal{F}$-transversality conditions instead of the universal local acyclicity conditions (cf. Lemma 4.2). However at that time, it is difficult to define the Artin conductor for a motive. Later in [YZ22], we observe that the Artin conductor for an étale constructible sheaf can be expressed in terms of the non-acyclicity class (cf. (3.9.1)). In the joint work [JY22] with Fangzhou Jin, we have successfully defined the Artin conductor of a constructible motive and formulate a quadratic version of the Grothendick-Ogg-Shafarevich formula (1.4.1).

Theorem 5.2 ([JY22, Theorem 1.3])**.** *Let $p : X \to \mathrm{Spec}(k)$ be a smooth and proper morphism with X connected, and let Z be a nowhere dense closed subscheme of X with open complementary U. Let $\mathcal{F} \in \mathbf{SH}_c(X)$ be a constructible motivic spectrum such that $\mathcal{F}|_U$ is dualizable. Then we have the following equality*

$$\chi(p_*\mathcal{F}) = p_*(\mathrm{rk}\mathcal{F} \cdot e(T_{X/k})) - a_Z^Q(\mathcal{F}) \quad \text{in} \quad \mathrm{GW}(k)[1/2]. \tag{5.2.1}$$

If k has characteristic different from 2 and X is odd-dimensional (for example, if $\dim X = 1$*), then one has*

$$\chi(p_*\mathcal{F}) = \mathrm{rk}\mathcal{F}_{\mathrm{et}} \cdot \chi(X/k) - a_Z^Q(\mathcal{F}) \quad \text{in} \quad \mathrm{GW}(k). \tag{5.2.2}$$

5.3. **Quadratic Milnor/conductor formula.** For the sphere spectrum $\mathbb{I}_k$, Levine, Lehalleur and Srinivas [LPS24] initiate the study of constructing a quadratic version of Bloch's conductor formula (and also Deligne's Milnor formula). They formulate a conjecture about a quadratic conductor formula for (isolated) homogeneous and quasi-homogeneous singularities. In order to handle the non-isolated case in positive characteristic, one has to consider the quadratic refinement of the Artin/Swan conductor.

Base on our work [JY22], now it is possible to answer (3) and (4) of 5.1 in the same time. Indeed, in [YZ22], we found that the conductor formula is a consequence of the functorial property of the non-acyclicity class. We could use the same strategy to formula and prove a quadratic conductor formula. For (4), if the dimension of the singular support of $\mathcal{F}$ equals $\dim X$, then we could be able to define a quadratic refinement for the characteristic cycle of $\mathcal{F}$ by using the method described in 3.8. Details of this part will appear in the near future (joint with Cisinski and Jin).

References

[Abb00] A. Abbes, *The Grothendieck-Ogg-Shafarevich formula for arithmetic surfaces*, Journal of Algebraic Geometry, 9 (2000): 529-576. ↑2, ↑4

[AS07] A. Abbes and T. Saito, *The characteristic class and ramification of an ℓ-adic étale sheaf*, Inventiones Math. 168 (2007): 567-612. ↑2, ↑6, ↑7, ↑8

[AS09] A. Abbes and T. Saito, *Analyse micro-locale ℓ-adique en caractéristique $p > 0$: Le cas d'un trait*, Publication of the Research Institute for Mathematical Sciences, 45-1 (2009): 25-74. ↑7

[AP18] T. Abe and D. Patel, *On a localization formula of epsilon factors via microlocal geometry*, Ann. K-Theory 3(3) 2018: 461-490. ↑6

[Abe21] T. Abe, *Ramification theory from homotopical point of view, I*, 2021, arXiv:2206.02401. ↑2, ↑3, ↑7

[Abe22] T. Abe, *On the Serre conjecture for Artin characters in the geometric case*, 2022, preprint. ↑9

[Bei07] A. Beilinson, *Topological $\mathcal{E}$-factors*, Pure Appl. Math. Q., 3(1, part 3) (2007):357-39. ↑3, ↑6

[Bei16] A. Beilinson, *Constructible sheaves are holonomic*, Sel. Math. New Ser. 22, (2016): 1797–1819. ↑3

[Blo87] S. Bloch, *Cycles on arithmetic schemes and Euler characteristics of curves*, Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), Proc. Sympos. Pure Math., vol. 46, Amer. Math. Soc., Providence, RI, (1987): 421–450. ↑2, ↑4

[Del73] P. Deligne, *La formule de dualité globale*, Exposé XVIII, pp.481-587 in SGA4 Tome 3: *Théorie des topos et cohomologie étale des schémas*, edited by M.Artin et al., Lecture Notes in Math.305, Springer, 1973. ↑4

[Del72] P. Deligne, *La formule de Milnor*, Exposé XVI, pp.197-211 in SGA7 II: *Groupes de Monodromie en Géométrie Algébrique*, Lecture Notes in Math. 340, Springer, 1973. ↑2

[Del72e] P. Deligne, *Les constantes des équations fonctionnelles des fonctions L* in Modular Functions of One Variable, II, Lecture Notes in Mathematics 349 Springer, Berlin-Heidelberg-New York, 1972. ↑5, ↑6

[Del11] P. Deligne, *Notes sur Euler-Poincaré: brouillon project*, 8/2/2011. ↑2, ↑3

[Gui22] Q. Guignard, *Geometric local ε-factors in higher dimensions*, Journal of the Institute of Mathematics of Jussieu, 21(6), (2022): 1887-1913. ↑6

[Hu15] H. Hu, *Refined characteristic class and conductor formula*, Math. Z., no.1-2, 281(2015): 571–609. ↑2

[JSY22] F. Jin, P. Sun and E. Yang, *The pro-Chern-Schwarz-MacPherson class in Borel-Moore motivic homology*, arXiv:2208.11989, 2022. ↑1

[JY21] F. Jin and E. Yang, *Künneth formulas for motives and additivity of traces*, Adv. Math. 376 (2021) 107446, 83 pages. ↑1

[JY22] F. Jin and E. Yang, *The quadratic Artin conductor of a motivic spectrum*, arXiv:2211.10985, 2022. ↑1, ↑2, ↑9, ↑12

[KS90] M. Kashiwara and P. Schapira, *Sheaves on manifolds*, Springer-Verlag, Grundlehren der Math. Wissenschaften, vol.292, Springer, Berlin (1990). ↑3, ↑7

[KS83] K. Kato and S. Saito, *Unramified class field theory of arithmetical surfaces*, Ann. of Math. 118 (1983):241-275. ↑5

[KS04] K. Kato and T. Saito, *On the conductor formula of Bloch*, Publications Mathématiques de l'IHÉS, (2004)100: 5-151. ↑2, ↑4

[KS08] K. Kato and T. Saito, *Ramification theory for varieties over a perfect field*, Ann. of Math. (2) 168 (2008), no. 1, 33–96. ↑1, ↑2, ↑3, ↑5

[KS12] K. Kato, and T. Saito, *Ramification theory for varieties over a local field*, Publications Mathematiques de IHES, (2013) 117: 1-178. ↑2, ↑3

[Lau83] G. Laumon, *Caractéristique d'Euler-Poincaré des faisceaux constructibles sur une surface*, Astérisque, 101-102 (1983): 193-207. ↑2, ↑4

[Lau87] G. Laumon, *Transformation de Fourier, constantes d'équations fonctionnelles et conjecture de Weil*, Publications Mathématiques de l'IHÉS, Volume 65 (1987): 131-210. ↑5, ↑6

[LZ22] Q. Lu and W. Zheng, *Categorical traces and a relative Lefschetz-Verdier formula*, Forum of Mathematics, Sigma, Vol.10 (2022): 1-24. ↑1, ↑6, ↑7

[LPS24] M. Levine, S. P. Lehalleur and V. Srinivas, *Euler characteristics of homogeneous and weighted-homogeneous hypersurfaces*, Adv. Math. 441 (2024) 109556, 86 pages. ↑12

[Ooe24] R. Ooe, *F-characteristic cycle of a rank one sheaf on an arithmetic surface*, arXiv:2402.06163. ↑4

[Org03] F. Orgogozo, *Conjecture de Bloch et nombres de Milnor*, Ann. Inst. Fourier 53 (2003), no. 6, 1739–1754. ↑4

[Pat12] D. Patel, *De Rham ε-factors*,Inventiones Mathematicae, Volume 190, Number 2, (2012): 299-355. ↑6

[Sai84] S. Saito, *Functional equations of L-functions of varieties over finite fields*, Journal of the Faculty of Science, the University of Tokyo. Sect. IA, Mathematics, Vol.31, No.2 (1984): 287-296. ↑5

[Sai93] T. Saito, *ε-factor of a tamely ramified sheaf on a variety*, Inventiones Math. 113 (1993): 389-417. ↑5

[Sai94] T. Saito, *Jacobi sum Hecke characters, de Rham discriminant, and the determinant of ℓ-adic cohomologies*, Journal of Algebraic Geometry, 3 (1994): 411–434. ↑6

[Sai17] T. Saito, *The characteristic cycle and the singular support of a constructible sheaf*, Inventiones Math. 207 (2017): 597-695. ↑1, ↑2, ↑3, ↑4, ↑5, ↑7, ↑9, ↑10

[Sai18] T. Saito, *On the proper push-forward of the characteristic cycle of a constructible sheaf*, Proceedings of Symposia in Pure Mathematics, volume 97, (2018): 485-494. ↑2, ↑5

[Sai21] T. Saito, *Characteristic cycles and the conductor of direct image*, J. Amer. Math. Soc. 34 (2021): 369-410. ↑2, ↑5, ↑6

[Sai22] T. Saito, *Cotangent bundles and micro-supports in mixed characteristic case*, Algebra & Number Theory, vol.16, no.2, (2022): 335-368. ↑4

[Tak19] D. Takeuchi, *Characteristic epsilon cycles of ℓ-adic sheaves on varieties*, arXiv:1911.02269, 2019. ↑4, ↑6

[Tak20] D. Takeuchi, *Symmetric bilinear forms and local epsilon factors of isolated singularities in positive characteristic*, arXiv:2010.11022, 2020. ↑4

[Tsu11] T. Tsushima, *On localizations of the characteristic classes of ℓ-adic sheaves and conductor formula in characteristic $p > 0$*, Math. Z. 269 (2011): 411-447. ↑2

[UYZ20] N. Umezaki, E. Yang and Y. Zhao, *Characteristic class and the ε-factor of an étale sheaf*, Trans. Amer. Math. Soc. 373 (2020): 6887-6927. ↑1, ↑5, ↑6, ↑7

[Vid09a] I. Vidal, *Formule du conducteur pour un caractère l-adique*, Compositio Math. 145 (2009): 687-717. ↑5

[Vid09b] I. Vidal, *Formule de torsion pour le facteur epsilon d'un caractère sur une surface*, Manuscripta math. 130 (2009): 21-44. ↑5

[XY23] J. Xiong and E. Yang, *Characteristic cycles and non-acyclicity classes for constructible etale sheaves*, https://www.math.pku.edu.cn/teachers/yangenlin/MF, 2023. ↑1, ↑10

[Y14] E. Yang, *Logarithmic version of the Milnor formula*, Asian J. Math. 21, No. 3 (2017). ↑4

[YZ21] E. Yang and Y. Zhao, *On the relative twist formula of ℓ-adic sheaves*, Acta. Math. Sin.-English Ser. 37 (2021): 73–94. ↑1, ↑5, ↑6, ↑7

[YZ22] E. Yang and Y. Zhao, *Cohomological Milnor formula and Saito's conjecture on characteristic classes*, arXiv:2209.11086, 2022. ↑1, ↑2, ↑6, ↑7, ↑8, ↑9, ↑10, ↑11, ↑12